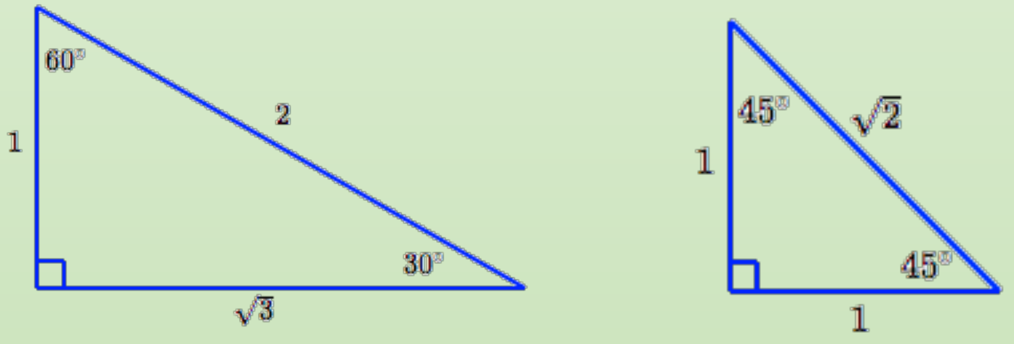
Quizz: on Thursday (last 20 mins. of studio):

- FTC (know how to apply it)
- integration by substitution
- IBP
- trig powers and product type integrals
- area between curves (*)

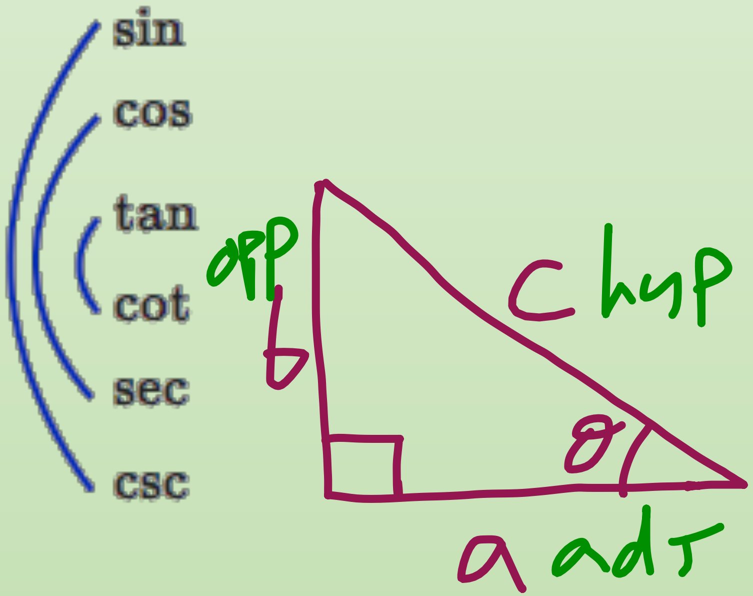
- **NO** trig subs. on this quiz
- make you know your special trig angles in **ALL** quadrants, e.g., $\cos/\sin\left(\frac{5\pi}{4}\right)$ or $\cos/\sin\left(\frac{7\pi}{4}\right)$

Review of Trigonometry

Special right triangles (ratio of sides):



Trig function inverse relationships diagram:



Rules to compute trig functions of right triangles:

SOHCAHTOA

Sine Opposite Hypotenuse | Cosine Adjacent Hypotenuse | Tangent Opposite Adjacent |

$a^2 + b^2 = c^2$

ex: $\sin(\theta) = b/c$

$\sec(\theta) = c/a$

$\tan(\theta) = b/a$

Credits for figures: https://www.onemathematicalcat.org/Math/Precalculus_obj/trigValuesSpecialAngles.htm

Form 1: When the integral contains a term of the form

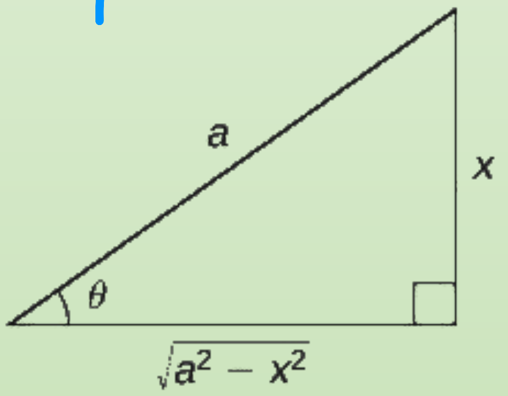
$$a^2 - x^2,$$

use the substitution:

$$\boxed{x = a\sin\theta}$$

Know how to get this triangle

$\sin\theta = \frac{x}{a}$



$\frac{dx}{d\theta} = a\cos\theta$

$$dx = a \cdot \cos\theta \, d\theta$$

$$\rightarrow a^2 - x^2 = a^2(1 - \sin^2\theta)$$
$$= a^2 \cdot \cos^2\theta$$

Credits for figure: https://math.libretexts.org/Bookshelves/Calculus
(*Book:* OpenStax -> Techniques of Integration -> Trigonometric Substitution – Section 7.3)

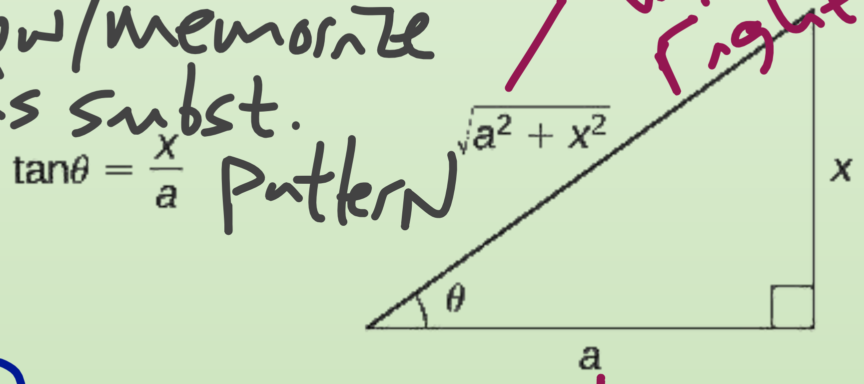
Form 2: When the integral contains a term of the form

$$a^2 + x^2,$$

use the substitution:

$$\boxed{x = a\tan\theta}$$

KNOW/memorize this subst. pattern



KNOW this value by right-▷ rules

(do NOT memorize)

recall:

$\tan^2\theta + 1 = \sec^2\theta$

$$dx = a \cdot \sec^2\theta \, d\theta$$

$$a^2 + x^2 = a^2 + a^2\tan^2\theta$$
$$= a^2(1 + \tan^2\theta)$$
$$= a^2 \cdot \sec^2\theta$$

Credits for figure: https://math.libretexts.org/Bookshelves/Calculus

(*Book:* OpenStax -> Techniques of Integration -> Trigonometric Substitution – Section 7.3)

Form 3: When the integral contains a term of the form

$$x^2 - a^2,$$

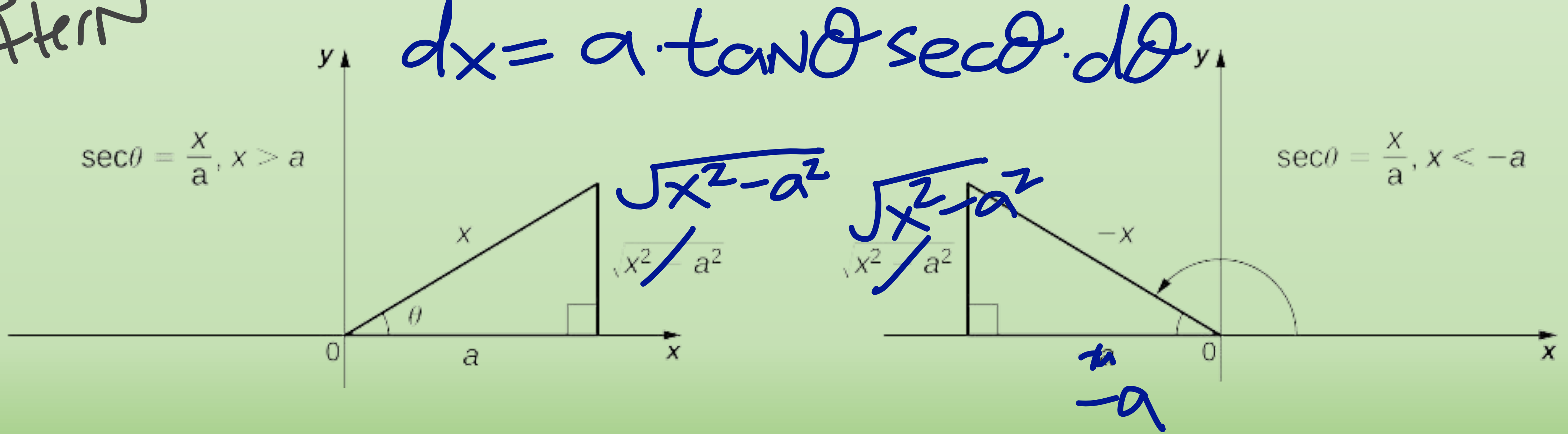
use the substitution:

$$\boxed{x = a\sec\theta}$$

know | memorize this type of subst pattern

$$x^2 - a^2 = a^2(\sec^2\theta - 1)$$
$$= a^2 \cdot \tan^2\theta$$

$$dx = a \cdot \tan\theta \sec\theta \cdot d\theta$$



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Extra problem: Evaluate the integral: $\int \frac{x}{\sqrt{x^2-3x+7}}dx = I$

$x^2-3x+7 \to x^2-2bx+b^2, \quad -2b=-3$

$b = 3/2$

$= (x-3/2)^2 + 7 - \frac{9}{4}$

$= (x-3/2)^2 + \frac{19}{4}$

$u = x - 3/2,$

$du = dx$

$I = \int \frac{x}{\sqrt{(x-3/2)^2+\frac{19}{4}}}$

for now write:

$a = \sqrt{\frac{19}{4}}$

$$= \int \frac{(x + 3/2)\,dx}{\sqrt{x^2 + a^2}} = I_1 + I_2,$$

where $I_1 = \int \frac{x\,dx}{\sqrt{x^2+a^2}}$ and

$$I_2 = \frac{3}{2} \int \frac{dx}{\sqrt{x^2+a^2}}$$

let's do the entire problem by a trig sub

$\underline{\text{trigsub}}$: $u^2+a^2 \longrightarrow u = a\cdot\tan\theta$

$du = a\sec^2\theta\, d\theta$

$$\sqrt{u^2+a^2} = \sqrt{a^2(1+\tan^2\theta)}$$

$$= a\cdot\sec\theta$$

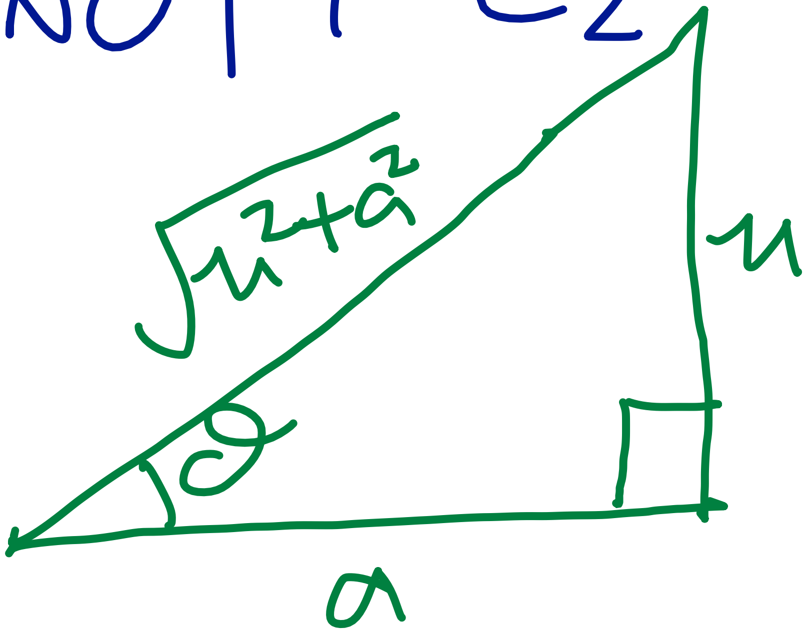
$$I = \int \frac{(a\cdot\tan\theta + 3/2)}{a\cdot\sec\theta} a\cdot\sec^2\theta\, d\theta$$

$$= \underbrace{\int a\cdot\tan\theta\sec\theta\, d\theta}_{I_1} + \frac{3}{2}\underbrace{\int \sec\theta\, d\theta}_{I_2}$$

$$I_1 = a \cdot \sec\theta + C_1$$

$$I_2 = \frac{3}{2} \ln|\sec\theta + \tan\theta| + C_2$$

$$\frac{u}{a} = \tan\theta$$



$$\sec\theta = \frac{\sqrt{u^2+a^2}}{a}$$

$$I = \sqrt{u^2+a^2} + \frac{3}{2}\ln\left|\frac{\sqrt{u^2+a^2}}{a} + \frac{u}{a}\right| + C$$

one more step: $u = x - 3/2$

$$u^2 + a^2 = x^2 - 3x + 7$$

$$I = \sqrt{x^2-3x+7} + \frac{3}{2}\ln\left|\sqrt{x^2-3x+7}\sqrt{\frac{4}{19}} + \frac{(x-3/2)}{\sqrt{\frac{19}{4}}}\right| + C$$

$$\int \cos^2\theta \, d\theta = \frac{1}{2}\int \left(\underbrace{1}_{} + \cos(2\theta) \right) d\theta$$

$$\int d\theta = \theta + C$$

Extra problem: Evaluate the integral: $\int e^{4x}\sqrt{1+4e^{2x}}\,dx = I$

use a u-sub first: $u = 2e^{x}$

$du = 2e^{x}dx$

$e^{4x}dx = \frac{1}{2}\left(\frac{u}{2}\right)^{3} du$

$I = \frac{1}{16}\int u^{3}\sqrt{1+u^{2}}\,du \quad (*)$

what happens if instead we do:

$$u = e^{2x}, \quad du = 2e^{2x}\,dx$$

$$e^{4x}\,dx = \frac{1}{2}\,du \cdot u$$

$$I = \frac{1}{2}\int u\sqrt{1+4u}\;du$$

(*) <u>trig sub</u>: $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$\sqrt{1+u^2} \doteq \sqrt{1+\tan^2\theta} = \sec\theta$$

$$I = \frac{1}{16}\int u^3\sqrt{1+u^2}\,du \quad (*)$$

recall that
$\tan^2\theta + 1$
$= \sec^2\theta$

$$I = \frac{1}{16}\int \tan^3\theta\cdot\sec^3\theta\,d\theta$$

$$= \frac{1}{16}\int(\sec^2\theta - 1)\sec^2\theta\cdot(\tan\theta\sec\theta)\,d\theta$$

Now, do another u-sub:

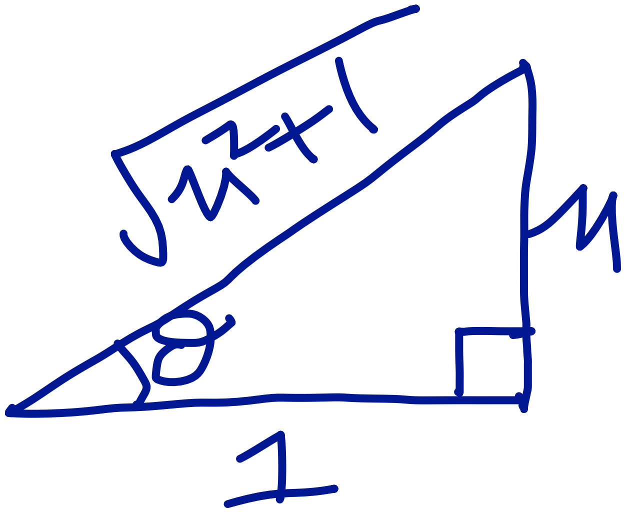
$u = \sec\theta$, $du = \tan\theta \cdot \sec\theta \, d\theta$

$$\rightarrow \underline{I} = \frac{1}{16} \int (u^4 - u^2)\, du$$

$$= \frac{1}{16} \left(\frac{u^5}{5} - \frac{u^3}{3} \right) + C$$

$$= \frac{1}{16} \left(\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} \right) + C$$

almost done: apply a right-Δ to rewrite in terms of u:



$$u = \tan\theta = \frac{u}{1}$$

$$\rightarrow \sec\theta = \sqrt{u^2+1}$$

So, $$I = \frac{1}{16}\left(\frac{(u^2+1)^{5/2}}{5} - \frac{(u^2+1)^{3/2}}{3}\right) + C$$

recall: $u = 2e^x$:

In total:

$$I = \frac{1}{16}\left(\frac{(4e^{2x}+1)^{5/2}}{5} - \frac{(4e^{2x}+1)^{3/2}}{3}\right) + C$$

Math 1552

Section 8.5:

The Method of Partial Fractions

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

When to Use Partial Fractions:

$$\int \frac{(x^3+2x+1)}{x^2+3}\,dx$$

Use the method of partial fractions to evaluate the integral of a *rational function* when:

- The degree of the numerator is *less than* that of the denominator.
- The denominator can be *completely factored* into linear and/or irreducible quadratic terms – *NO complex numbers in this class!*

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a "1", factor it out.

$$\int \frac{x\,dx}{2x^2+4x+10}$$

→ pull out the factor of two

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.

Quick refresher on polynomial long division

Question: What do you when asked to evaluate this integral? $\int \frac{x^3 - 2x^2 - 4}{x-3} dx = I$

Short answer: Observe that $x^3 - 2x^2 - 4 = (x-3)(x^2+x+3) + \mathbf{5}$ (How?)

$$
\begin{array}{r}
x^2 + x + 3 \\
x-3 \,\big)\, x^3 - 2x^2 + 0x - 4 \\
-(x^3 - 3x^2) \\
\hline
x^2 + 0x \\
-(x^2 - 3x)
\end{array}
\qquad
\begin{array}{r}
\hline
3x - 4 \\
-(3x - 9) \\
\hline
5
\end{array}
$$

(This standard method works for denominator polynomials of degree larger than one.)

What this shows is that:

$$x^3-2x^2-4=(x-3)(x^2+x+3)+5$$

$$I=\int(x^2+x+3)\,dx+5\int\frac{dx}{x-3}$$

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.
3. Factor the denominator completely into linear and/or irreducible quadratic terms.

Partial Fractions Procedure:

4. For each linear term of the form $(x-a)^k$, you will have *k* partial fractions of the form:

$$\boxed{\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}}$$

(Note: if *k*=1, there is only one fraction to handle, etc.)